ON DEFORMATIONS AND MODULI
OF GENUS 2 FIBRATIONS

Hurşit Önsiper

Abstract

In this paper, we investigate the moduli of surfaces of general type admitting genus 2 fibrations with irregularity $q = g_b + 1$, where $g_b \geq 2$ is the genus of the base. We prove that smooth fibrations are parametrized by a unique component in the moduli space. The same result applies to nonsmooth fibrations with special values of g_b. In the general case, we give a bound on the dimension of the corresponding connected components.

Surfaces admitting genus 2 fibrations appear as important examples of special surfaces of general type and there has been considerable effort to analyze their structure and moduli. In particular, in ([S1], [S2]) Seiler obtained substantial results on the structure of the moduli spaces of surfaces admitting genus 2 albanese fibrations, via detailed study of deformations of ruled surfaces covered by surfaces admitting genus 2 fibrations.

In this paper, we take up the same problem from a more elementary point of view; we apply results and methods from the theory of curves to study surfaces admitting genus 2 fibrations over curves of genus $g_b \geq 2$ with irregularity $q = g_b + 1$. The basic idea is to reduce the problem to the study of deformations of the base curve and of the fibers. To this end, the first main ingredient is the fundamental result due to Siu and Beauville (cf. [B2], [C]) to the effect that the question of whether a surface admits a fibration over a curve of genus ≥ 2 is of completely topological nature. In the same direction, fibrations obtained from harmonic maps were applied by Jost and Yau ([J-Y1], [J-Y2], [J-Y3]) to yield related results which complement and strengthen the results obtained by algebro-geometric techniques. Combining these facts with the results of Xiao on the structure of genus 2 fibrations ([X1]), the study of the moduli spaces of the surfaces under consideration readily reduces to problems amenable to techniques from the theory of curves.

The case of smooth fibrations satisfying $q = g_b + 1$ is easier to settle and in the first part of the paper we prove

Theorem 1 : *Given $g_b \geq 2$ and K^2, χ satisfying $K^2 = 8\chi = 8(g_b - 1)$. Then in the moduli space of surfaces of general type with these invariants K^2, χ, there exists a unique*

AMS Subject Classification (1991). 14J10, 14J15.

connected component of dimension $3g_b - 1$ parametrizing smooth genus 2 fibrations with $q = g_b + 1$.

In the rest of the paper, we deal with nonsmooth fibrations. In this part of the paper, the results of Xiao ([X1]) on universal genus 2 fibrations and the work by Namba ([N]) on families of holomorphic maps from compact Riemann surfaces into the projective line play crucial role.

We fix $1 \leq K^2, \chi$ to satisfy $K^2 < 8\chi$ and the equality $K^2 = \lambda\chi(X) + (8-\lambda)(g_b - 1)$ for some $g_b \geq 2$, where the slope λ satisfies $\lambda = 7 - \frac{6}{d}$, $d \geq 2$. With this notation, we have the following partial result which complements results obtained by Seiler, because the surfaces we consider do not admit genus 2 albanese fibrations.

Theorem 2 :
(i) For $d \leq 5$ and $g_b \geq 2$, depending on K^2, χ there exists a finite number of integers $n_i \geq 2$, $i = 1,..,k$ and for each n_i a connected component of dimension $\leq 2n_i + 2g_b - 4$ parametrizing nonsmooth genus 2 fibrations with $q = g_b + 1$.
(ii) For $d \geq 6$ there exists a component parametrizing nonsmooth genus 2 fibrations only if $g_b \geq (d-6)\frac{d^2}{24}\prod_{p|d}(1 - \frac{1}{p^2}) + 1$. For $d \geq 7$ and $g_b = (d-6)\frac{d^2}{24}\prod_{p|d}(1 - \frac{1}{p^2}) + 1$, we have a unique component of dimension 1.

Throughout the paper we will consider only minimal surfaces over **C** and fibrations are always assumed to be connected. We adapt the following standard notation.

$\chi(?), e(?), K_?, p_g(?), q(?)$ are the holomorphic Euler characteristic, the topological Euler characteristic, the canonical class, the geometric genus and the irregularity of ?, respectively.
X is a smooth compact minimal surface admitting a fibration $X \to S$ with base genus $g_b \geq 2$, fiber genus 2 and $q = g_b + 1$.
$\pi_1(?)$ is the fundamental group of ?.
$\mathcal{M}_g$ is the moduli space of curves of genus g.

We first observe that any deformation X' of X also admits a fibration $X' \to S'$ with $q = g_b + 1$. In fact, since $\pi_1(X') = \pi_1(X)$, by Beauville's criterion ([B2]) we have a fibration $X' \to S'$ with $g(S') \geq g_b$. If $g(S') > g_b$, then as $g_b + 1 = q(X') \geq g(S')$ we have $g(S') = g_b + 1$ and again by Beauville's criterion we see that X is fibered by the albanese map too. This however is impossible, because then the first fibration on X can not be connected. Therefore, to understand the moduli of genus 2 fibrations with $q = g_b + 1$, we

must determine
(i) when the fibration $X' \to S'$ on a deformation X' of X is of fiber genus 2, and
(ii) when two such genus 2 fibrations can be deformed to each other.

For smooth fibrations both of these questions can be easily answered. In fact we will prove a general result for analytic fiber bundles of base and fiber genera ≥ 2. It is well known that such a fiber bundle with fiber F becomes trivial over an unramified Galois base extension with group $\mathcal{G} = Image(\pi_1(S) \to Aut(F))$. In this notation, we have

Proposition 1 : *Let $X \to S$ be a fiber bundle of fiber genus $g \geq 2$ and base genus $g_b > (g+1)/2$. Then each deformation of X admits an analytic fiber bundle structure with the same trivializing group $\mathcal{G}$ as X.*

Proof :

We know by ([J-Y1], Lemma 7.1) that fiber bundles of genus g with base of genus g_b deform to such bundles. In what follows, we will prove that the trivializing group is fixed in a family of fiber bundles parametrized by a connected base.

For a given $\mathcal{G}$, we take $g' = |\mathcal{G}|(g_b - 1) + 1$ and we let $T_g, \mathcal{C}_g$ (resp. $T_{g'}, \mathcal{C}_{g'}$) be the Teichmüller space of Riemann surfaces of genus g (resp. g') and the universal curve on T_g (resp. on $T_{g'}$). We consider the submanifolds $T_g^{\mathcal{G}}$ of T_g parametrizing curves F with $\mathcal{G} \subset Aut(F)$ and $T'_{g'}$ of $T_{g'}$ parametrizing curves of genus g' admitting $\mathcal{G}$ as a fixed point free automorphism group. We note that $dim(T'_{g'}) = 3g_b - 3$, because $\mathcal{G}$ being a quotient of $\pi_1(S)$, any curve of genus g_b admits an etale cover of genus g'. We let $\mathcal{F} \to T_g^{\mathcal{G}} \times T'_{g'}$ be the quotient by $\mathcal{G}$ of $\mathcal{C}_g \times \mathcal{C}_{g'}$ restricted to $T_g^{\mathcal{G}} \times T'_{g'}$. For a point $p \in T_g^{\mathcal{G}} \times T'_{g'}$ we consider the following diagram

$$\begin{array}{ccccc}
0 \to \mathcal{T}_p(T_g^{\mathcal{G}} \times T'_{g'}) & \to \mathcal{T}_p(T_g \times T_{g'}) & \to \mathcal{T}_p(T_g^{\mathcal{G}} \times T'_{g'}) \\
\downarrow & \downarrow & \downarrow \\
0 \to H^1(\mathcal{F}_p, \Theta) & \to H^1(F \times S, \Theta) & \to H^1(\mathcal{F}_p, \Theta)
\end{array}$$

where the vertical arrows are the corresponding Kodaira-Spencer maps and the far right horizontal arrows are the trace maps induced by the action of $\mathcal{G}$. As the middle vertical arrow is an isomorphism, we immediately check that the Kodaira-Spencer map of the family $\mathcal{F} \to T_g^{\mathcal{G}} \times T'_{g'}$ is bijective. Hence this family is versal and as $H^0(\mathcal{F}_p, \Theta) = 0$ for all fibers $\mathcal{F}_p$ of the family, it is the universal deformation for any of its fibers.

Therefore, in a given deformation $\mathcal{X} \to \mathcal{S}$ of fiber bundles of genus 2, locally over the base the trivializing group is fixed. Now, since $g_b > (g+1)/2$, we have $K_X^2 = 8(g_b - 1)(g - 1) >$

$4(g-1)^2$ and therefore none of the surfaces $\mathcal{X}_s$ admits two different genus g fibrations unless $\mathcal{X}_s$ is a trivial product ([X1], Proposition 6.4). As trivial products deform to trivial products ([J-Y], Cor. 6.1), it follows that over a connected base the trivializing group is fixed.$\square$

From the proof of Proposition 1, it also follows that any two fiber bundles as in the statement of the proposition deform to each other. Therefore, we see that in the moduli space of surfaces with invariants $K^2 = 8\chi = 8(g_b - 1)(g-1)$, analytic fiber bundles with fiber genus g and base genus g_b, if they exist, form a single connected component $\mathcal{M}$. The existence for $g = 2$ case is proved in the following elementary lemma.

Lemma 2 : *Given $g_b \geq 2, K^2 = 8(g_b - 1)$ and $\mathcal{G}$ as listed in ([X1], p. 30), there exists a genus 2 analytic fiber bundle with trivializing group $\mathcal{G}$.*

Proof :

Given $\mathcal{G}$, we take a curve F of genus 2 with $\mathcal{G} \subset Aut(F)$.

We first observe that $\mathcal{G}$ is generated by two elements $\{g_1, g_2\}$ if the hyperelliptic involution σ on F is not in $\mathcal{G}$ and by the four elements $\{g_1, g_2, \sigma g_1, \sigma g_2\}$ if $\sigma \in \mathcal{G}$ where $\{g_1, g_2\}$ are coset representatives of $\langle \sigma \rangle$ in $\mathcal{G}$ ([X1], p. 30). Therefore, for a curve S of genus ≥ 2 writing $\pi_1(S) = \langle a_j, b_j : \prod [a_j, b_j] = e \rangle$ we can define a surjective homomorphism

$\pi_1(S) \to \mathcal{G}$

by $a_j \mapsto g_j, b_j \mapsto g_j^{-1}$ (resp. $a_j \mapsto g_j, b_j \mapsto \sigma g_j$) for $j = 1, 2$ if $\sigma \notin \mathcal{G}$ (resp. $\sigma \in \mathcal{G}$) and sending all other a_j, b_j to identity. Via this homomorphism we construct an etale cover $S_1 \to S$ with Galois group $\mathcal{G}$. The quotient surface $X = F \times S_1/\mathcal{G}$ admits a fibration $X \to S$ of the type desired.$\square$

For genus 2 fiber bundles with $q = g_b + 1$, $\mathcal{G} = \mathbf{Z}_2$ and the hyperelliptic involution σ of the fiber is not in $\mathcal{G}$. Taking $\mathcal{G}$ as described, the following lemma completes the proof of Theorem 1.

Lemma 3 : *Let $\mathcal{G} \neq \{e\}$ be one of the groups listed in ([X1], p. 30). Then $\mathcal{M} \cong \mathcal{M}_2^{\mathcal{G}} \times \mathcal{M}_{g_b}$, where $\mathcal{M}_2^{\mathcal{G}}$ is the moduli space of curves of genus 2 admitting $\mathcal{G}$ as a group of automorphisms.*

Proof :

Two fibers of the universal family $\mathcal{F}$ of the proof of Proposition 1 are isomorphic if and only if they lie on $(x_1, y_2), (x_2, y_2) \in T_2^{\mathcal{G}} \times T_{g'}^{'}$, where $x_1 \cong x_2 \pmod{\Gamma_{\mathcal{G}}}$ and $y_1 \cong y_2 \pmod{\Gamma_{g_b}}$, $\Gamma_{\mathcal{G}}$ being the modular group determined by $\mathcal{G}$. Hence $\mathcal{M} \cong T_2^{\mathcal{G}}/\Gamma_{\mathcal{G}} \times T_{g'}^{'}/\Gamma_{g_b} \cong \mathcal{M}_2^{\mathcal{G}} \times \mathcal{M}_{g_b}$.
$\square$

For nonsmooth fibrations we first answer question (i) posed at the beginning of the paper, for arbitrary fiber genus $g \geq 2$.

Lemma 4 :
(i) A surface of general type X admits, if any, a unique nonsmooth fibration $X \to S$ with $g_b \geq g$ satisfying $q = g_b + 1$.
(ii) If X admits such a fibration, then so does any deformation X' of X.

Proof :

(i) By the criterion given in ([X1], Proposition 6.4), it suffices to check that $K(X)^2 > 4(g-1)^2$.

For a fibration as in the statement of the lemma, we have

$K^2 = \lambda\chi(X) + (8-\lambda)(g_b - 1)$,

where $\lambda \geq 4$ ([X2], p. 459, Corollary 1). Therefore, as

$\chi(X) = (g_b - 1)(g-1) - deg(R^1 f_*(\mathcal{O}_X)) > (g_b - 1)(g-1)$

we get $K(X)^2 > 4\chi(X) > 4(g-1)^2$ as required.

(ii) Existence of the fibration on X' is clear from the discussion at the beginning of the paper. To see that fiber genus is g, by the uniqueness of the fibration it suffices to observe that the fiber genus is locally constant over the base of a deformation. But this is a consequence of ([Ö], Proposition 2) which in this context states that for a given deformation $\mathcal{X} \to \mathcal{Y}$ of X, locally around any $y \in \mathcal{Y}$ the deformation factors over a surface $\mathcal{C}$ to give a deformation of the fibers of $X \to S$. $\Box$

As a consequence of Lemma 4, we see that in the moduli space of surfaces with invariants K^2, χ, if there exist points corresponding to surfaces with nonsmooth genus 2 fibrations satisfying $q = g_b + 1$, then they form connected components. We will state our results on the existence and the dimension of such components with respect to the values of the invariant d of the fibrations under consideration. First we recall the following basic facts.

1) Given a genus 2 fibration $X \to S$ with invariant d, the fibration is said to be of type (E, d) if the fixed part of the associated jacobian fibration is the elliptic curve E.

2) For each pair (E, d) where E is an elliptic curve and $d \geq 2$, there exists a genus 2 fibration $X(E, d) \to M(d)$ of type (E, d) over the modular curve $M(d)$ such that any fibration $X \to S$ of this type is obtained as the minimal desingularization of the pull back $f^*(X(E, d))$ of $X(E, d)$ via a surjective morphism $f : S \to M(d)$ ([X1], Corollaire on p.46).

Proof of Theorem 2 :

In the construction of the universal fibration $X(E,d) \to M(d)$ given in ([X1], p.42), letting $u_1(z) = (z_1, 0), u_6(z) = \frac{1}{d}(z_1, 1)$ vary, we obtain a family $\mathcal{X}(d) \to M(d) \times \mathbf{A}^1$ where for $a \in \mathbf{A}^1$, $\mathcal{X}(d)|_{M(d)\times a} \cong X(E_a, d)$, E_a being the elliptic curve with j-invariant $j(E_a) = a$.

Given a deformation $\mathcal{X} \to \mathcal{Y}$ of surfaces admitting fibrations as considered in this part of the paper, then around any $y \in \mathcal{Y}$ the given deformation is induced from a deformation of the fibration $\mathcal{X}_y \to S_y$. Hence it follows that, locally on $\mathcal{Y}$, $\mathcal{X} \to \mathcal{Y}$ is obtained via a morphism F into $M(d) \times \mathbf{A}^1$ where F composed with the projection onto $\mathbf{A}^1$ maps y to the j-invariant $j(E_y)$. Therefore, in the moduli space of surfaces with fixed K^2, χ, the dimension of any component parametrizing surfaces with nonsmooth genus 2 fibrations satsfying $q = g_b + 1$ is bounded by $m(d) + 1$ where $m(d)$ the dimension of the space of holomorphic maps from compact Riemann surfaces of genus g_b into the modular curve $M(d)$.

To prove part (i) of Theorem 2, we note that for $d \leq 5$, $M(d) \cong \mathbf{P}^1$ and we apply the results in ([N]) to get
a) in a deformation $\mathcal{X} \to \mathcal{Y}$ of X over a connected base, the degree of the maps into $M(d) \cong \mathbf{P}^1$ inducing the fibration $\mathcal{X}_y \to S_y$ is independent of y ([N], Lemma 3.3.3),
b) for a fixed $n \geq 2$, the space of holomorphic maps of degree n from compact Riemann surfaces of genus g_b into $\mathbf{P}^1$ (modulo $Aut(\mathbf{P}^1)$) is of dimension $2n + 2g_b - 5$ ([N], Theorem 3.4.17).

On the otherhand, as a consequence of the restriction on $K^2(X), \chi(X)$ if X is obtained from $X(E,d)$ via a map of degree n with prescribed ramification divisor, it follows that for a given pair K^2, χ there exists, if any, only a finite number of integers $n_i \geq 2$ appearing as the degree of maps from Riemann surfaces of genus g_b into $M(d)$ inducing surfaces with these given invariants. Combining this observation with (a) and (b) above, part (i) of Theorem 2 follows.

(ii) If there exists a nonsmooth genus 2 fibration of type (E,d) over a curve S of genus g_b, then we have a surjective map $S \to M(d)$ and hence
$g_b \geq genus(M(d)) = (d-6)\frac{d^2}{24}\prod_{p|d}(1 - \frac{1}{p^2}) + 1$ for $d \geq 6$.
Furthermore, if we have $d \geq 7$ and $g_b = g(M(d))$, then since $g(M(d)) \geq 2$, it follows that the map $S \to M(d)$ is an isomorphism. Therefore, the moduli space of such fibrations is $\mathbf{A}^1$, the modulus map being given by the j-invariant of E if X is of type (E,d). This completes the proof of Theorem 2. $\square$

References

[B1] A. Beauville, L'inégalité $p_g \geq 2q - 4$ pour les surfaces de type général, Bull. Soc. Math. Fr. 110, 344 - 346 (1982).

[B2] A. Beauville, appendix to F. Catanese, Moduli and classification of irregular Kähler manifolds (and algebraic varieties) with albanese general type fibration, Invent. Math. 104, 263 - 289 (1991).

[C] F. Catanese, Moduli and classification of irregular Kähler manifolds (and algebraic varieties) with albanese general type fibration, Invent. Math. 104, 263 - 289 (1991).

[J-Y1] J. Jost, S. T. Yau, Harmonic mappings and Kähler manifolds, Math. Ann. 262, 145 - 166 (1983).

[J-Y2] J. Jost, S. T. Yau, Harmonic mappings and algebraic varieties over function fields, Amer. J. Math. 115, 1197 - 1227 (1993).

[J-Y3] J. Jost, S. T. Yau, Harmonic maps and Kähler geometry, in Prospects in Complex Geometry, LNM 1468, Sringer-Verlag (1991), 340 - 370 .

[Ö] H. Önsiper, A note on degenerations of fibered surfaces, Indag. Math. 8(1), 115 - 118 (1997).

[S1] W. K. Seiler, Moduli spaces for special surfaces of general type, in C. Bajaj (ed.), Algebraic geometry and its applications, Springer-Verlag (1994), 161 - 172.

[S2] W. K. Seiler, Moduli of surfaces of general type with a fibration by genus two curves, Math. Ann. 301, 771 - 812 (1995).

[X1] G. Xiao, Surfaces fibree en courbes de genre deux, LNM 1137, Springer-Verlag (1985).

[X2] G. Xiao, Fibered algebraic surfaces with low slope, Math. Ann. 276, 449 - 466 (1987).

Department of Mathematics
Middle East Technical University
06531 Ankara, Turkey

e-mail : hursit@rorqual.cc.metu.edu.tr